

The T_4 and G_4 constructions of Costas arrays

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Abstract

We examine two particular constructions of Costas arrays known as the Taylor variant of the Lempel construction, or the T_4 construction, and the variant of the Golomb construction, or the G_4 construction. We connect these constructions with the concept of Fibonacci primitive roots, and show that under the Extended Riemann Hypothesis the T_4 and G_4 constructions are valid infinitely often.

1 Introduction

A Costas array is an $N \times N$ array of dots with the properties that one dot appears in each row and column, and that no two of the $N(N-1)/2$ line segments connecting dots have the same slope and length. It is clear that a permutation f of $\{1, 2, \dots, N\}$, from the columns to the rows (i.e. to each column x we assign exactly one row $f(x)$), gives a Costas array if and only if for $x \neq y$ and $k \neq 0$ such that $1 \leq x, y, x+k, y+k \leq N$, then $f(x+k) - f(x) \neq f(y+k) - f(y)$.

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Costas arrays were first considered by Costas [4] as permutation matrices with ambiguity functions taking only the values 0 and (possibly) 1, applied to the processing of radar and sonar signals. The use of Costas arrays in radar is summarized in [11, §5.2]. Costas arrays are also used in the design of optical orthogonal codes for code division multiple access (CDMA) networks [14], and in the construction of low-density parity-check (LDPC) codes [1].

Let us briefly recall some known constructions on Costas arrays. One can find more details in the survey papers of Golomb and Taylor [10, 9], Drakakis [5], Golomb and Gong [8]. In the following, p is taken to be a prime and q a prime power. The known general constructions for $N \times N$ Costas arrays are the Welch construction for $N = p - 1$ and $N = p - 2$, the Lempel construction for $N = q - 2$, and the Golomb construction for $N = q - 2$, $N = q - 3$. Moreover, if $q = 2^k$, $k \geq 3$, the Golomb construction works for $N = q - 4$. The validity of the Welch and Lempel constructions is proved by Golomb in [6]. The Golomb constructions for $N = q - 3$ and $N = 2^k - 4$ depend on the existence of (not necessarily distinct) primitive elements α and β in \mathbb{F}_q such that $\alpha + \beta = 1$. The existence of primitive elements α and β in \mathbb{F}_q such that $\alpha + \beta = 1$ was proved by Moreno and Sotero in [15]. (Cohen and Mullen give a proof with less computational checking in [2]; more recently, Cohen, Oliveira e Silva, and Trudgian proved [3] that, for all $q > 61$, every non-zero element in \mathbb{F}_q can be written as a linear combination of two primitive roots of \mathbb{F}_q .)

Among these algebraic constructions over finite fields, there are the T_4 variant of the Lempel construction for $N = q - 4$ when there is a primitive element α in \mathbb{F}_q such that $\alpha^2 + \alpha = 1$, and the G_4 variant of the Golomb construction for $N = q - 4$ when there are two primitive elements α and β such that $\alpha + \beta = 1$ and $\alpha^2 + \beta^{-1} = 1$. Through the study of primitive elements of finite fields, Golomb proved in [7] that q must be either 4, 5 or 9, or a prime $p \equiv \pm 1 \pmod{10}$ in order for the T_4 construction to apply. Note that this is a necessary but not sufficient condition (for example $p = 29$). In the same paper, Golomb also proved that the values of q such that the G_4 construction occurs are precisely $q = 4, 5, 9$, and those primes p for which the T_4 construction occurs and which satisfy either $p \equiv 1 \pmod{20}$ or $p \equiv 9 \pmod{20}$.

In this paper, we connect the T_4 and G_4 constructions with the concept of Fibonacci primitive roots. We show, in Theorems 1 and 2, that under the Extended Riemann Hypothesis (ERH) there are infinitely many primes such that T_4 and G_4 can apply. We conclude with some observations and questions about trinomials of primitive roots.

2 Fibonacci primitive roots

The T_4 construction requires a primitive root α such that

$$\alpha^2 + \alpha = 1. \tag{1}$$

To investigate the nature of solutions to (1) we recall the notion of a *Fibonacci primitive root*, or *FPR*. We say that g is a FPR modulo p if $g^2 \equiv g + 1 \pmod{p}$. Shanks and Taylor [18] proved a similar statement to that which we give below.

Lemma 1. *If g is a FPR modulo p , then $g - 1$ is a primitive root modulo p that satisfies (1), and vice versa.*

Proof. It is clear that g satisfies $g^2 \equiv g + 1 \pmod{p}$ if and only if $g - 1$ satisfies (1): all that remains is to check that g and $g - 1$ are primitive. Suppose first that g is a FPR modulo p . Then, since $g(g - 1) \equiv 1 \equiv g^{p-1}$, we have

$$(g - 1)^n \equiv g^{p-n-1} \pmod{p},$$

Note that, as n increases from 1 to $p - 1$, g^{p-n-1} generates \mathbb{F}_p , since g is primitive. Hence $g - 1$ is a primitive root modulo p . The converse is similarly proved. \square

Let $F(x)$ denote the number of primes $p \leq x$ that have at least one FPR. Shanks [17] conjectured that under ERH, $F(x) \sim C\pi(x)$, where $\pi(x)$ is the prime counting function, and where $C \approx 0.2657\dots$. Lenstra [12] proved Shanks' conjecture; a proof also appears in Sander [16]. We therefore have

Theorem 1. *Let $T(x)$ be the number of primes $p \leq x$ for which p satisfies the T_4 construction. Then, under the Extended Riemann Hypothesis*

$$T(x) \sim \frac{27}{38}\pi(x) \prod_{p=2}^{\infty} \left(1 - \frac{1}{p(p-1)}\right) \sim (0.2657\dots)\pi(x).$$

Unconditionally, it seems difficult to show that there are infinitely many primes that have a FPR. Phong [13] has proved some results about a slightly more general class of primitive roots. For our purposes, [13, Cor. 3] implies that if $p \equiv 1, 9 \pmod{10}$ such that $\frac{1}{2}(p - 1)$ is prime then there exists (exactly) one FPR modulo p . This does not appear, at least to the authors, to make the problem any easier!

We turn now to the G_4 construction, which requires two primitive roots α, β such that

$$\alpha + \beta = 1, \quad \alpha^2 + \beta^{-1} = 1.$$

Since we require that $p \equiv 1, 9 \pmod{20}$ we are compelled to ask: how many of these primes have a FPR? We can follow the methods used in [12, §8], and also examine Shanks' discussion in [17, p. 167]. Since we are now only concerned with $p \equiv 1, 9 \pmod{20}$ we find that the asymptotic density should be $\frac{9}{38}A$, where $A = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p(p-1)}\right) \approx 0.3739558138$ is Artin's constant. This leads us to

Theorem 2. *Let $G(x)$ be the number of primes $p \leq x$ for which p satisfies the G_4 construction. Then, under the Extended Riemann Hypothesis*

$$G(x) \sim \frac{9}{38}\pi(x) \prod_{p=2}^{\infty} \left(1 - \frac{1}{p(p-1)}\right) \sim (0.08856\dots)\pi(x).$$

3 Conclusion

One can show that, for $p > 7$ there can be no primitive root α modulo p that satisfies $\alpha + \alpha^{-1} \equiv 1 \pmod{p}$. (Suppose there were: then $\alpha^2 + 1 \equiv \alpha \pmod{p}$ so that $\alpha^3 + \alpha^2 + 1 \equiv \alpha^2 \pmod{p}$ whence $\alpha^3 \equiv -1 \pmod{p}$. Hence $\alpha^6 \equiv 1 \pmod{p}$ — a contradiction for $p > 7$.) From this, it follows that $x^{p-2} + x - 1$ is never primitive over \mathbb{F}_p for $p > 7$.

Consider the following question: given $1 \leq i \leq j \leq p-2$, let $d(i, j)$ denote the density of primes for which there is a primitive root α satisfying $\alpha^i + \alpha^j \equiv 1 \pmod{p}$. The above comments show that $d(1, p-2) = 0$; Theorem 1 shows that under ERH, $d(1, 2) \approx 0.2657$. What can be said about $d(i, j)$ for other prescribed pairs (i, j) ? In the case $i = j$, we have $2\alpha^i \equiv 1 \pmod{p}$ and thus $\alpha^i = \frac{p-1}{2}$. In particular, if $(i, p-1) = 1$ then it is equivalent to ask for the density of primes such that $\frac{p-1}{2}$ is a primitive root modulo p . We have not been able to find a reference for this in the literature, though computational evidence seems to suggest that this value should be close to Artin's constant $0.37395\dots$

When $i \neq j$, it is easy to see that $d(2, \frac{p-1}{2} + 1) = d(1, 2)$. Therefore, under ERH the trinomial $x^{\frac{p-1}{2}+1} + x^2 - 1$ is primitive over \mathbb{F}_p for infinitely many primes p . More generally, we can show that for $p > 3i$ there does not exist a primitive root α such that $\alpha^{\frac{p-1}{2}+i} + \alpha^{\frac{p-1}{2}+2i} \equiv 1 \pmod{p}$, and thus $d(\frac{p-1}{2} + i, \frac{p-1}{2} + 2i) = 0$. Similarly, $d(i, 2i + \frac{p-1}{2}) = 0$. Indeed, if $\alpha^i - \alpha^{2i} \equiv 1 \pmod{p}$ for a primitive α , we obtain $\alpha^{3i} \equiv \alpha^{2i} - \alpha^i \equiv -1 \pmod{p}$. Hence we can show that if $p > 6i$ there is no primitive element α such that $\alpha^i + \alpha^{2i + \frac{p-1}{2}} \equiv 1 \pmod{p}$. Using the same arguments as before, we can also show that $d(i, p-1-i) = 0$ for any prefixed i .

References

- [1] S. C. Chae and Y. O. Park, *Low complexity encoding of improved regular LDPC codes*, 2004 IEEE 60th Vehicular Technology Conference (VTC2004-Fall, Los Angeles, CA, September 26-29, 2004), vol. 4, 2004, 2535–2539.
- [2] S. D. Cohen and G. L. Mullen, *Primitive elements in finite fields and Costas arrays*, Appl. Algebra Engrg. Comm. Comput. **2** (1991), no. 1, 45–53.
- [3] S. D. Cohen, T. Oliveira e Silva, and T. S. Trudgian. *A proof of the conjecture of Cohen and Mullen on sums of primitive roots*. Math. Comp., to appear.
- [4] J. P. Costas, *Medium constraints on sonar design and performance*, Proceedings of EASCON (Washington, D.C., September 29–October 1, 1975), 68A–68L.
- [5] K. Drakakis, *A review of Costas arrays*, J. Appl. Math. **2006** (2006), 1–32.
- [6] S. W. Golomb, *Algebraic constructions for Costas arrays*, J. Combin. Theory Ser. A **37** (1984), no. 1, 13–21.
- [7] S. W. Golomb, *The T_4 and G_4 constructions for Costas arrays*, IEEE Trans. Inform. Theory **38** (1992), no. 4, 1404–1406.

- [8] S. W. Golomb and G. Gong. *The status of Costas arrays*, IEEE Trans. Inform. Theory, **53** (2007), no. 11, 4260–4265.
- [9] S. W. Golomb and H. Taylor, *Constructions and properties of Costas arrays*, Proc. IEEE **72** (1984), no. 9, 1143–1163.
- [10] S. W. Golomb and H. Taylor, *Two-dimensional synchronization patterns for minimum ambiguity*, IEEE Trans. Inform. Theory **28** (1982), no. 4, 600–604.
- [11] N. Levanon and E. Mozeson, *Radar signals*, John Wiley & Sons, 2004.
- [12] H. W. Lenstra, *On Artin’s conjecture and Euclid’s algorithm in global fields*, Inventiones math. **42** (1977), 201–224.
- [13] B. M. Phong, *Lucas Primitive Roots*, Fibonacci Quart. **29** (1991), no. 1, 66–71.
- [14] S. V. Maric, M. D. Hahm, and E. L. Titlebaum, *Construction and performance analysis of a new family of optical orthogonal codes for CDMA fiber-optic networks*, IEEE Trans. Commun. **43** (1995), no. 234, 485–489.
- [15] O. Moreno and J. Sotero, *Computational approach to Conjecture A of Golomb*, Congr. Numer. **70** (1990), 7–16.
- [16] J. W. Sander. *On Fibonacci primitive roots*, Fibonacci Quart. **28** (1990), no. 1, 79–80.
- [17] D. Shanks. *Fibonacci primitive roots*, Fibonacci Quart. **10** (1972), no. 2, 163–181.
- [18] D. Shanks and L. Taylor. *An observation on Fibonacci primitive roots*, Fibonacci Quart. **11** (1973), no. 2, 159–160.